Scientific Visualization, 2025, volume 17, number 1, pages 26 - 35, DOI: 10.26583/sv.17.1.03

Isotropy in Local Computer Geometry

A.V. Tolok[1], N.B. Tolok[2]
V.A. Trapeznikov Institute of Management Problems of the Russian Academy of Sciences, Moscow, Russia

[1] ORCID: 0000-0002-7257-9029, tolok_61@mail.ru
[2] ORCID: 0000-0002-5511-4852, nat_tolok@mail.ru

Abstract

The article discusses the property of isotropy in local computer geometry. The basic principles of applying such a property in the representation of computer data about the domain of a function are demonstrated using the example of a function of two arguments. The scope of application of the isotropic property in algebraic transformations, data packing and encoding is considered. The effect of isotropy in algebraic transformations is given using the example of the product of two functions. The formation of the domain of local functions of the paraboloid surface for describing a circle is illustrated on the basis of the domain of local functions for describing the surface for a square. The possibility of computer representation of the domain of local functions by a single graphical M-image is analyzed.

Keywords: Isotropy, local computer geometry, Functional voxel modeling, M-image, computer graphics, data packing, encoding.

Problem statement

The purpose of the conducted research was to study the possibility and methods of application of *isotropy of a local function* which is the one of the important properties in local computer geometry, underlying the fundamental difference of this section of geometry from the section of differential geometry [1-5]. Such approach significantly expands the range of applied problems solved by local geometric modeling. The basic means of representing information in local computer geometry [6, 7] are local geometric characteristics for the neighborhood of points on a given region with dimension m, describing a homogeneous unit vector $\vec{n}$. The components of such a homogeneous unit vector $(n_1, \dots, n_{m+1})$ determine the local function $n_1 x_1 + \dots + n_m x_m + n_{m+1} = 0$ at each point of the region X_m. The local function, in turn, describes a linear law duplicating at a given point any other law specified by the analytical representation $F(X_m) = 0$ (Fig. 1).

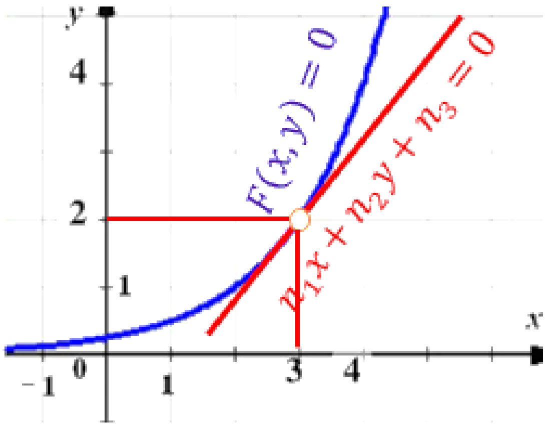


Fig.1. Duplication of the law *F(x)=0* by a local function at a point

It is known that the derivative of the function gives the slope of the tangent line to the curve at this specific point, which means that it can be applied as a linear law at this point, replacing the law given by the original function. This property leads to a simplification of analytical calculations at this point, which allowed Isaac Newton and Gottfried Wilhelm Leibniz to develop the theory of differential and integral calculus. In fact, it is easy to show that tangential differential calculus can be attributed to a special case of local geometric calculus in general, and especially in the considered case of local computer geometry, since only modern computer technologies have made it possible to process large amounts of data for the development of such calculus.

1. Isotropy of the local function

For simplicity and clarity, let us consider the principle of isotropy for the two-dimensional case. Let's turn to Figure 2 and assume that the local function passes through a point orthogonally to the tangential direction.

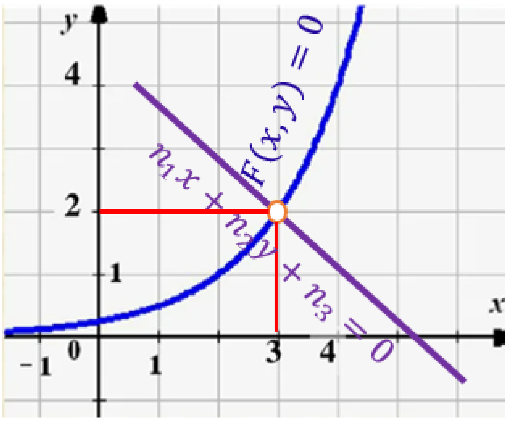


Fig.2. Duplication of the law $F(x, y) = 0$ by an orthogonal local function at a point

It can be argued that such a local function also duplicates the law of the function $F(x, y) = 0$ (or explicitly $y = f(x)$) as well as the tangential local function, since the following relation is preserved:

$$y = f(x) = -\frac{n_1}{n_2}x - \frac{n_3}{n_2} = -3\frac{n_1}{n_2} - \frac{n_3}{n_2} = 2. \tag{1}$$

It is interesting that for any local functions describing the equation of a straight line passing through a selected point, the property of duplicating the function $y = f(x)$ is preserved. We can say that a "bundle" of local functions at a single point can duplicate any function $y = f(x)$ passing through this point. That is, for a local function at the point, the principle of isotropy is observed: *the uniformity of ratios* of the local function arguments *in all orientations*.

Using the example of a function of three variables $F(x, y, z) = 0$, we will show that for any arbitrarily given components n_1, n_2, n_3 of a homogeneous vector of a local function, it is always possible to calculate the fourth component n_4, leading to an unambiguous determination of the range of the original function $F(x, y, z) = 0$.

Let's consider an example of describing the range of function values for the zero contour "square". We will describe such a range using R-functional modeling tools:

$$z = (1 - x^2) + (1 - y^2) - \sqrt{(1 - x^2)^2 + (1 - y^2)^2}. \tag{2}$$

Function (2) provides a zero value along the contour of a square with a side equal to two units and centered at the origin. For consideration, we will select a 4x4 region centered at the origin.

In order to obtain the domain of tangential local functions, i.e. the domain of tangent planes, it is sufficient to create a regular grid covering for the given domain. To define a triangular element of the plane, for each vertex we add two neighboring vertices shifted along the

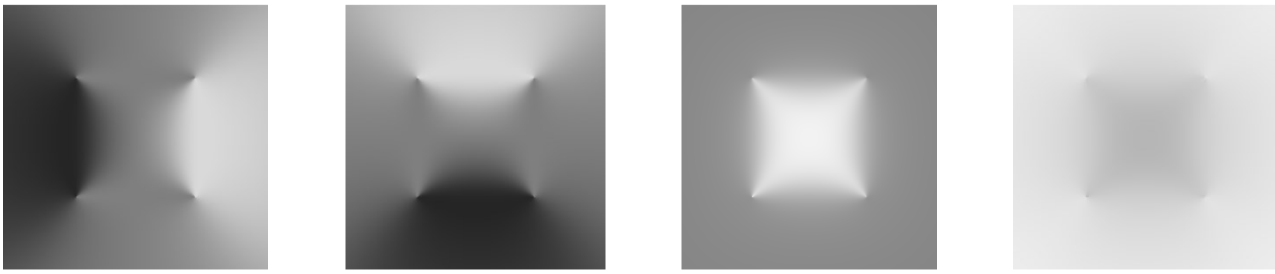
Ox and Oy axes by the grid step. For each of these three vertices we calculate z using formula (2) and formulate a local function using the linear operator [6,7]:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = a_1 x + a_2 y + a_3 z + a_4 = 0. \tag{3}$$

Reducing the components to a uniform unit vector and matching the monochrome color palette gradation P on the M-image, we obtain:

$$n_i = \frac{a_i}{\sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}}, \qquad M_i = \frac{P(1 + n_i)}{2}, \ P = 256. \tag{4}$$

Figure 3 shows M-images that display on a computer the range of all four characteristics n_1, n_2, n_3, n_4, respectively.



M_1 M_2 M_3 M_4

Fig.3. M-images describing the range of tangential local functions for a zero square contour

Figure 4 shows the positive range of values of the function (2) in monochrome, and the blue colour represents negative values of this function.

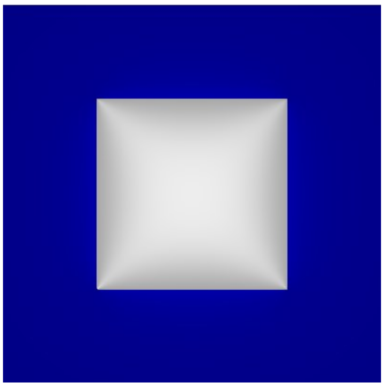
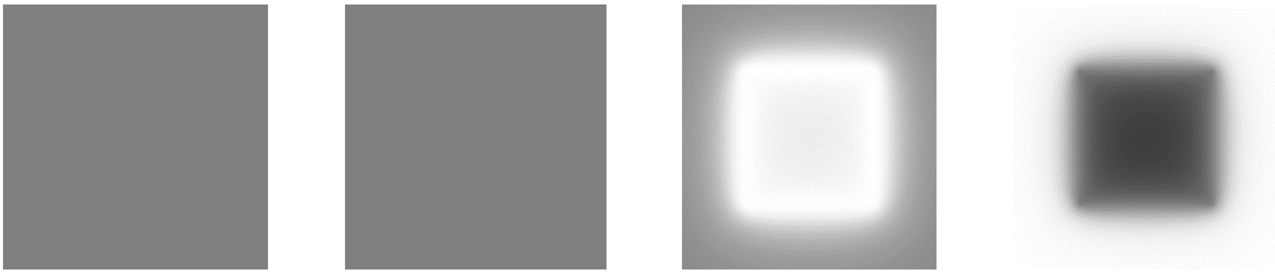


Fig.4. Positive and negative value ranges of the function (2) superimposed on the image M_3

To begin with, let's model a situation where the local function is constant and horizontal at each point, i.e. $a_1 = 0, a_2 = 0, a_3 = 1$. At the same time, the first two images - M_1 and M_2 - possess a gray color corresponding to the zero value, and the image M_3 takes on a white color corresponding to a unit value. Now we have to to determine the fourth image M_4, which carries basic information about the function (2). To do this, at the current point $\boldsymbol{i}$ with coordinates (x_i, y_i), calculate z_i using formula (2) and find the coefficient a_4:

$$a_4 = -a_1 x_i - a_2 y_i - a_3 z_i. \tag{5}$$

We will reduce the obtained components a_i to the components of a homogeneous unit vector n_i, and then to the correspondence with the monochrome color palette P according to formula (4), we will obtain the M-images shown in Figure 5.



M_1 M_2 M_3 M_4

Fig.5. M-images describing the range of local functions for the zero square contour generated by the last component a_4 - for $a_1 = 0, a_2 = 0, a_3 = 1$

The first two M-images keep the color constancy equal to the value 127 (gray), obtained after normalization, for the zero value of the cosine of the angle deviation with a reference to the Ox and Oy axes. During normalization, the third M-image M_3 is influenced by the values of the norm, since the numerator in this case is not zero as in the first two cases.

Let us consider the value of z at each point of a given region using a local function according to the formula

$$z_i = -\frac{n_1}{n_3}x_i - \frac{n_2}{n_3}y_i - \frac{n_4}{n_3}. \tag{6}$$

We will display the range of negative values of z_i in blue, making sure that it matches Figure 4 (Fig. 6).

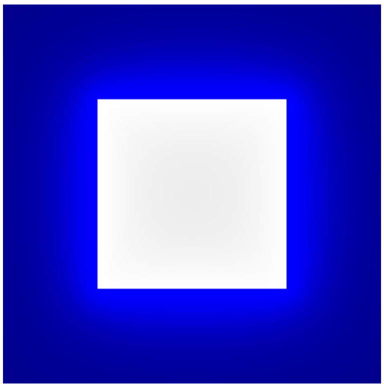
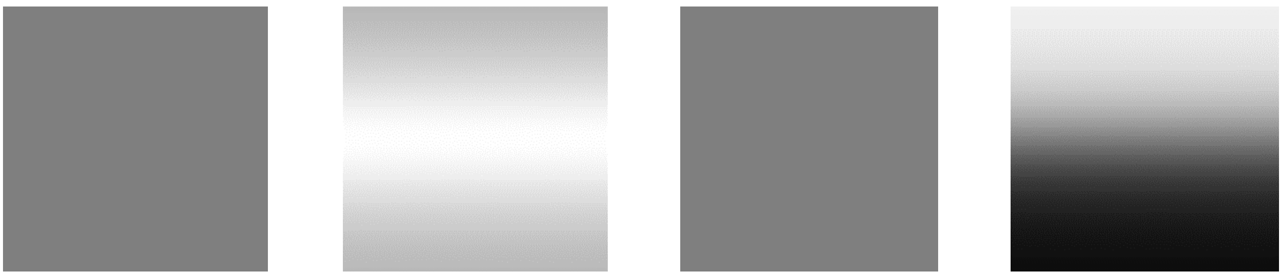


Fig. 6. Image of the positive and negative range of z values of local functions

The orientation of the planes described by local functions that are orthogonal to the Oy and Ox axes, respectively, is shown in Figures 7 and 8. Here we can observe how the component a_4 influences the color of the M-image of the corresponding component a_i, equal to one.



M_1 M_2 M_3 M_4

Fig.7. M-images describing the range of local functions for the zero square contour generated by the last component a_4 - for $a_1 = 0, a_2 = 1, a_3 = 0$.

M_1 M_2 M_3 M_4

Fig.8. M-images describing the range of local functions for the zero square contour generated by the last component a_4 - for $a_1 = 1, a_2 = 0, a_3 = 0$.

2. Reducing the computer representation of the function range to a single M-image with a digital key encryption element

The conducted studies have shown that the developed approach allows using a single image M_4, with known components a_1, a_2, a_3 expressed by constants, to obtain the domain of local functions describing the domain of function (2) by means of the proposed key encryption algorithm. The idea is to use the procedure of replacing the entire image containing the constant color at each point with one numerical value. For example, in the special case considered in Figure 5, the M-images M_1, M_2 and M_3 can be represented by the numbers 0,0,1, and cases represented by Figures 7 and 8 contain the numerical combinations 0,1, 0 and 1,0,0 in M-images. Thus, for any of the selected combinations with the available values of a_1, a_2 and a_3, it is sufficient to calculate the value of a_4 at each point of the corresponding image M_4. In this case, the combination of a_1, a_2 and a_3 is the key to make the choice between three considered images M_4. Let's consider the algorithm for calculating a_4.

We have already determined that the value of n_4 at the point of the image M_4 is calculated by the formula

$$n_4 = \frac{2M_4 - 256}{256} \tag{7}$$

With the values of components $a_1 = 0, a_2 = 0, a_3 = 1$, coefficient a_4 can be expressed from formula (4). In the considered case

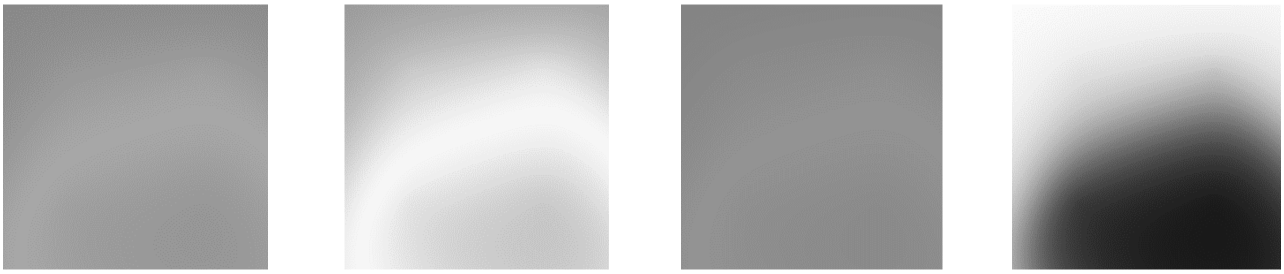
$$n_4 = \frac{a_4}{\sqrt{1 + a_4^2}}. \tag{8}$$

As a result, we obtain:

$$a_4 = \frac{n_4}{\sqrt{1 - n_4^2}}. \tag{9}$$

It follows that each of the three M-images M_4 in Figures 5, 7, 8, together with its numeric key, contains sufficient initial graphical information to obtain the same function domain (2). This means that it is possible to completely restore the Functional Voxel model shown in Figure 3 if any of the M-images are available together with a numerical key. Note also that as the dimensionality of the considered space of function arguments increases, the number of zero components will increase correspondingly, i.e. formula (9) remains universal, while the index for a_i and n_i increases.

Now let's consider the more complicated problem by assigning different values of integer type to three constants (for example, $a_1 = 2, a_2 = 6, a_3 = 1$). Using formula (5), we obtain the value of a_4 (Fig. 9).



M_1 M_2 M_3 M_4

Fig.9. M-images describing the range of function (2) for $a_1 = 2, a_2 = 6, a_3 = 1$

Тогда Let's try to express a_4 again, given that all three components a_1, a_2, a_3 are known to be some numeric values other than zero. Then

$$n_4 = \frac{a_4}{\sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}}, \tag{10}$$

that leads to

$$a_4 = n_4 \frac{\sqrt{a_1^2 + a_2^2 + a_3^2}}{\sqrt{1 - n_4^2}}, \tag{11}$$

where a_1, a_2, a_3 – a complex key encoding for an M-image M_4.

Figure 10 depicts the result of calculating the negative and positive ranges of z values for local functions composed by M-images represented in Figure 9.

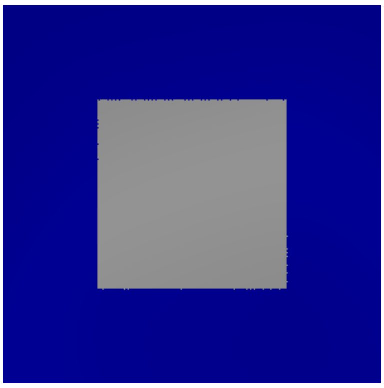


Fig.10. Positive (gray) and negative (blue) ranges of z for local functions with the key 2,6,1.

It is not difficult to show that any three images from this composition of the functional voxel model, replaced by constants, make it possible to generate one of the M-images to describe the function domain.

3. Application of the isotropic property in computational operations on homogeneous vectors

In [6,7,8], computational operations on local functions using an M-image representation of a given domain are considered. A typical example of the isotropy occurs when obtaining the result of component-by-component multiplication of the values of two different functions represented by homogeneous vectors at points in a given area.

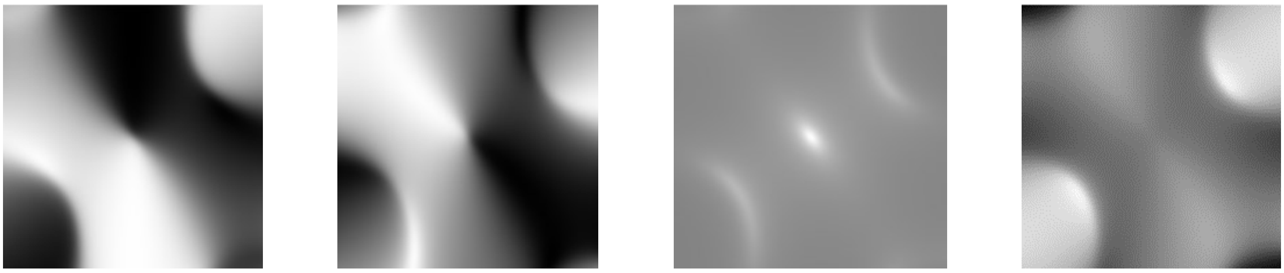
As an example, let us consider different types of functions:

- trigonometric f-function (Fig. 11)

$$z^f = 5(y sin\pi x + x^2 cos\pi y) \tag{12}$$

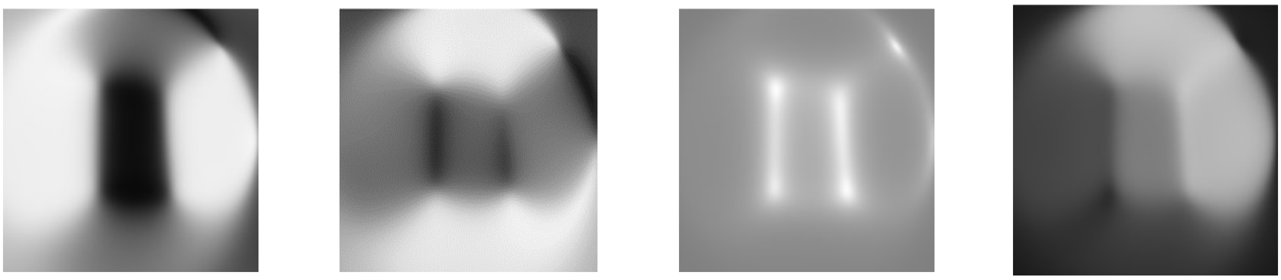
and exponential g -function (Fig. 12)

$$z^g = (x-1)e^{-[x^2+(y+1)^2]} + 10(0{,}2x - x^3 - y^5)e^{-(x^2+y^2)} + e^{-\frac{(x^2+y^2)}{3}}. \tag{13}$$



M_1 M_2 M_3 M_4

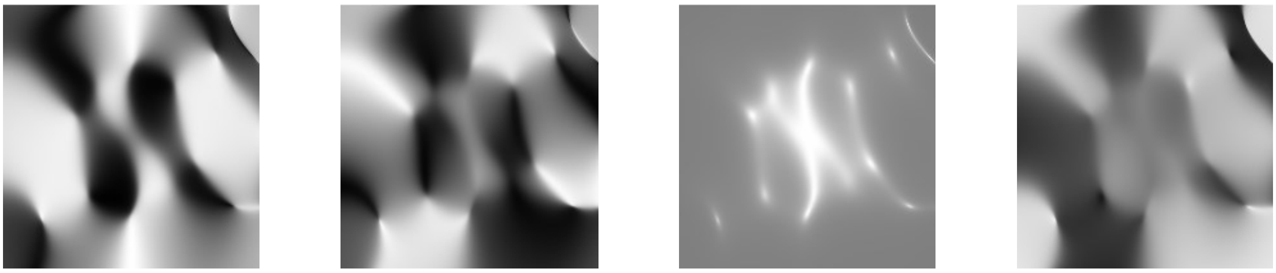
Fig. 11. M-images representing the domain of local functions for the trigonometric function (12)



M_1 M_2 M_3 M_4

Fig.12. M-images representing the domain of local functions for the exponential function (13)

The benchmark for comparing those results will be images of local functions obtained directly by the multiplication of functions (12) and (13), shown in Figure 13.



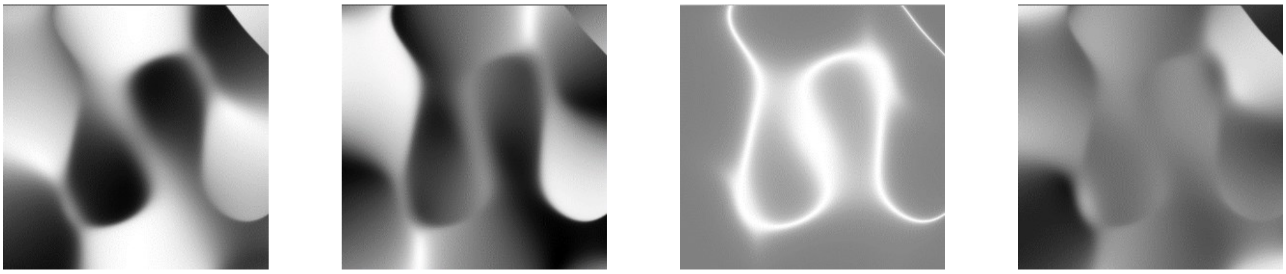
M_1 M_2 M_3 M_4

Fig.13. M-images representing the domain of local functions for the product of functions (12) and (13)

In [6,7,8], the solution to the problem of determining the product based on local functions of a homogeneous vector is proposed to be expressed as follows:

$$\begin{gathered}
z^{fg}=\left(-\frac{n_1^f}{n_3^f}x-\frac{n_2^f}{n_3^f}y-\frac{n_4^f}{n_3^f}\right)\times\left(-\frac{n_1^g}{n_3^g}x-\frac{n_2^g}{n_3^g}y-\frac{n_4^g}{n_3^g}\right),\\
z^{fg}=\left(-\frac{n_1^f}{n_3^f}x-\frac{n_2^f}{n_3^f}y-\frac{n_4^f}{n_3^f}\right)\times z^g,\\
n_1^{fg}=n_1^f z^g, n_2^{fg}=n_2^f z^g, n_3^{fg}=n_3^f,\ n_4^{fg}=n_4^f z^g,\\
z^{fg}=-\frac{n_1^{fg}}{n_3^{fg}}x-\frac{n_2^{fg}}{n_3^{fg}}y-\frac{n_4^{fg}}{n_3^{fg}}.
\end{gathered}\tag{14}$$

If in the previous description, Figure 13 demonstrates tangential local functions obtained by linear approximation using formulas (3) and (4), then the result of calculations using formula (14) is not tangential (Fig. 14), but the z^{fg}-surface differs in the 14th decimal place at each point of the considered domain.



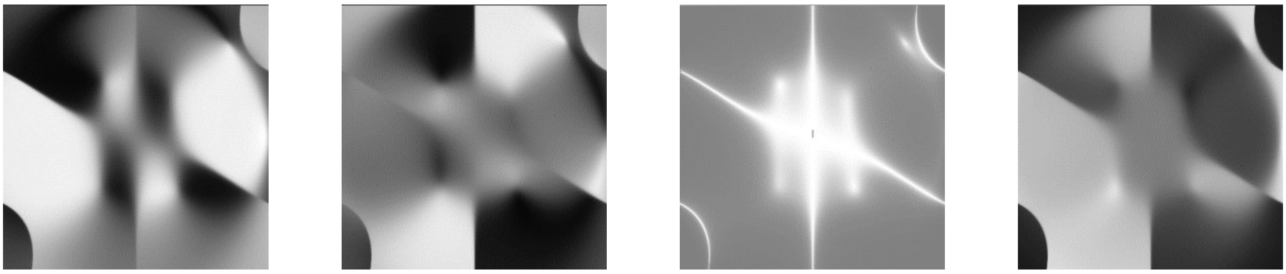
M_1 M_2 M_3 M_4

Fig. 14. M-images describing the domain of local functions according to formula (14)

It is clear that there is another solution that allows you to express the components:

$$z^{fg} = \left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right) \times \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right),$$
$$z^{fg} = z^f \times \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right),$$
$$n_1^{fg} = n_1^g z^f, n_2^{fg} = n_2^g z^f, n_3^{fg} = n_3^g,\ n_4^{fg} = n_4^g z^f,$$
$$z^{fg} = -\frac{n_1^{fg}}{n_3^{fg}}x - \frac{n_2^{fg}}{n_3^{fg}}y - \frac{n_4^{fg}}{n_3^{fg}}. \tag{15}$$

In this case, the M-images will, of course, show a different picture (Fig. 15), however, the z^{fg} surface will coincide with the previously considered cases.



M_1 M_2 M_3 M_4

Fig.15. M-images describing the domain of local functions according to formula (15)

Thus, due to the existence of the isotropic property, the commutative property of multiplication is preserved in the arithmetic operations of local geometry.

4. Transition to a tangential local function

The question arises: is it possible to obtain tangential local functions from any M-image representation of a function?

Let us consider an example describing the process of generating the set of M-images represented at Figure 3 directly from the image M_4 displayed at Figure 9. To do this, we transform the color value of each current point of the image M_4 into the value of the fourth component of the homogeneous unit vector n_4 in terms of expression (6) and express the component a_4 using given conditions $a_1 = 2, a_2 = 6, a_3 = 1$:

$$n_4 = \frac{2M_4 - P}{P}, \qquad a_4 = n_4\frac{\sqrt{a_1^2 + a_2^2 + a_3^2}}{\sqrt{1 - n_4^2}} = n_4\frac{\sqrt{2^2 + 6^2 + 1^2}}{\sqrt{1 - n_4^2}} = n_4\frac{\sqrt{41}}{\sqrt{1 - n_4^2}}. \tag{16}$$

The expression of color gradation according to formula (4) leads to the restoration of all four M-images of Figure 9:

$$n_1 = \frac{2}{\sqrt{41 + a_4^2}}, n_2 = \frac{6}{\sqrt{41 + a_4^2}}, n_3 = \frac{1}{\sqrt{41 + a_4^2}}, n_1 = \frac{a_4}{\sqrt{41 + a_4^2}}, \tag{17}$$

$$M_i = \frac{P(1+n_i)}{2},\ P = 256.$$

The resulting M-image representation provides the z-value of the function (2) at the corresponding points of the domain:

$$z = -\frac{n_1}{n_3}x - \frac{n_2}{n_3}y - \frac{n_4}{n_3}. \quad (18)$$

At this stage, taking into account the discreteness of the M-images, we should apply a linear approximation of the function domain by restoring ordered triples of vertices of a rectangular grid, following the calculations on the basis of formulas (3) and (4).

Conclusions

The conducted studies have shown the wide possibilities of computing tools and forms of representation of a multidimensional domain of a complex function on a computer when using the components of a homogeneous vector to describe a single point in space. The revealed isotropic property significantly expands the methods of such representation and allows to pack the computer data into a single M-image, reducing the remaining images to constant values, which helps to solve the problems of graphical encryption of the object geometry. It is shown that the isotropy simplifies the automation of algebraic and arithmetic calculations over functions, which allows the implementation of complex computational structures, such as R-functions [6], etc., using local functions. This paper also presents a method for transitioning to the traditional tangential position of local functions for performing differential and integral calculus [9-11].

Acknowledgements

The research was carried out within the framework of the scientific program of the National Center for Physics and Mathematics, direction No. 9 "Artificial intelligence and big data in technical, industrial, natural and social systems".

References

1. Egorov A.I. Ordinary differential equations with applications. Fizmatlit, Moscow, 2003. 384 pp.
2. Romanko V.K. Differential equations and the calculus of variations. Moscow: Laboratory of Basic Knowledge, 2000. 344 pp.
3. Stepanov V.V. The Course in Differential Equations. M.: Editorial URSS, 2004. 472 pp.
4. Konev V.V. Partial differential equations: Study guide / Tomsk Polytechnic University.: https://portal.tpu.ru/SHARED/k/KONVAL/notes/Partial.pdf.
5. Krasnov M.L., Kiselev A.I., Makarenko G.I. Ordinary differential equations: Problems and examples with detailed solutions: Study guide. Moscow: LENAND, 2019. – 256 pp. (Higher mathematics in problems and exercises)
6. Tolok A.V. (2022) Local computer geometry. Study guide. IPR Media, Moscow– 147 pp.
7. Tolok, A.V. (2016) Functional Voxel Method in Computer Modeling. Fizmatlit, Moscow – 112 pp.
8. Alexey Tolok, Natalya Tolok. Arithmetic in Functional-Voxel Modeling (2022). Scientific Visualization 14.3: 107 - 121, DOI: 10.26583/sv.14.3.08
9. Tolok A.V., Tolok N.B. Differentiation and Integration in Functional Voxel Modeling // CONTROL SCIENCES. 2022. № 5. P. 51-57.
10. Tolok A.V., Tolok N.B. The Functional Voxel Method Applied To Solving a Linear First-Order Partial Differential Equation with Given Initial Conditions // CONTROL SCIENCES. 2023. № 6. P. 65-71.
11. A.V. Tolok, N.B. Tolok. Functional-Voxel Modeling of The Cauchy Problem (2024). Scientific Visualization, 2024, v. 16, N 1, p. 105 - 111, DOI: 10.26583/sv.16.1.09